SOME SIMPLIFICATIONS IN THE DESCRIPTION OF THE MOTION OF SOFT SOIL

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The system of equations which has been proposed in earlier papers [1,2] for the representation of the dynamics of soft soils is very complicated. However, for the study of various particular problems it is possible to simplify this system. It has been thought, therefore, desirable to establish a way of characterizing any concrete problem so that the possible simplifications for each case and the simplification in the system of equations is taken advantage of. In this paper a complete classification of any possible problem is given for the system of equations and a simplified version of the system for each class of problem is derived.

1. The equations derived in [1,2] have the form

$$\rho\left(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right)=\rho F_i^{\,e}-\frac{\partial p}{\partial x_i}+\frac{\partial S_{ij}}{\partial x_j} \tag{1.1}$$

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v_i)}{\partial x_i}=0,\qquad p=f(\theta,\theta_*),\qquad \theta\equiv 1-\frac{\rho_0}{\rho},\qquad \theta_*\equiv 1-\frac{\rho_0}{\rho_*}$$

$$\frac{\partial\theta_*}{\partial t}+v_i\frac{\partial\theta_*}{\partial x_i}=\left(\frac{\partial\theta}{\partial t}+v_i\frac{\partial\theta}{\partial x_i}\right)e(\theta-\theta_*)\,e\left(\frac{\partial\theta}{\partial t}+v_i\frac{\partial\theta}{\partial x_i}\right)$$

$$G\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}-\frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right)=\frac{DS_{ij}}{Dt}+\lambda S_{ij}$$

$$\frac{DS_{ij}}{Dt}=\frac{\partial S_{ij}}{\partial t}+v_k\frac{\partial S_{ij}}{\partial x_k}-\Omega_{ik}S_{jk}-\Omega_{jk}S_{ik},\qquad \Omega_{ij}=\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}-\frac{\partial v_j}{\partial x_i}\right)$$

$$\lambda=\frac{2GW-F'(p)(\partial p/\partial t+v_i\partial p/\partial x_i)}{2F(p)}\,e\,[I_2-F(p)]\times$$

$$\times e\left[2GW - F'(p)\left(\frac{\partial p}{\partial t} + v_i\frac{\partial p}{\partial x_i}\right)\right]$$

$$W \equiv \frac{1}{2} S_{ij}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \qquad I_2 \equiv \frac{1}{2} S_{ij}S_{ij}$$

In these equations certain empirical functions appear which serve to characterize the soil

$$p = f(\theta, \theta_*), \quad I_2 = F(p), \quad G = G(\theta_*) \tag{1.2}$$

A method for determining these functions by means of dynamic and static experiments has been indicated in [1,2] and some results of such experiments are contained in [3,4].

The fundamental problem of soil mechanics, arising from this system of equations (1.1), consists of the following: Let there be given a region Ω, filled with the medium, initially at rest. On part of the boundary of this region the stresses $\mathbf{P}_n$ are given and on the remainder of the boundary, the deformations $\mathbf{u}$ are known as functions of the coordinates of the boundary and of time. It is required to determine the ensuing motion of the medium and the distribution of stresses.

The given boundary functions $\mathbf{P}_n$ and $\mathbf{u}$ may always be represented in the form

$$\mathbf{P}_n = \sigma_0 \Pi_n\left(\frac{x_i}{l}, \frac{t}{t_0}\right), \qquad \mathbf{u} = V_0 t_0 \mathbf{U}\left(\frac{x_i}{l}, \frac{t}{t_0}\right) \tag{1.3}$$

where Π_n and $\mathbf{U}$ are dimensionless functions of dimensionless arguments, so that boundary conditions may be inserted into the mathematical formulation of the problem by means of the parameters σ_0, V_0, t_0 and l, which characterize the magnitude of the stresses, velocities, times and linear dimensions. Different classes of problems will be defined for the same boundary conditions through differences in the numerical values of these parameters. In order that the classification may be carried out, it is necessary that these parameters be supplemented by additional ones contained in the systems of equations (1.1). To express properly these parameters, we observe that relation (1.2) may always be represented in the form

$$p = Kf(\theta, \theta_*), \qquad I_2 = \sigma_*^2 F(p/\sigma_*), \qquad G = G_0 g(\theta_*)$$

$$\underset{\substack{\theta\to 0\\ \theta_*\to \min\theta_*}}{f(\theta, \theta_*) \to \theta}, \qquad \underset{p/\sigma_*\to 0}{F(p/\sigma_*) \to 1}, \qquad \underset{\theta_*\to \min\theta_*}{g(\theta_*) \to 1} \tag{1.4}$$

We introduce the notation

$$S_\infty = \lim_{p/\sigma_*\to\infty} \sqrt{\sigma_*^2 F(p/\sigma_*)} = \sigma_* \sqrt{F(\infty)} \tag{1.5}$$

Experiments show that

$$K \sim G_0, \qquad \sigma_* \ll G_0 \tag{1.6}$$

In addition, one may deduce that S_∞ is of the same order as G_0 or smaller, i.e.

$$S_\infty \lesssim G_0 \tag{1.7}$$

Thus, the parameters necessary for the description also include K, G_0, σ_*, S_∞, ρ_0 so that the complete system of constants essential to define the problem parametrically takes on the form

$$\sigma_0,\ V_0,\ t_0,\ l,\ K,\ G_0,\ \sigma_*,\ S_\infty,\ \rho_0 \tag{1.8}$$

It is necessary to mention that in cases for which the boundary conditions are given only in terms of the stresses or only the deformations, the system will contain only one of the two parameters, σ_0 or V_0 and there will be another relation connecting them.

The classification will reduce, in the end, to establishing certain estimates and inequalities for dimensionless combinations, formed from the parameters in (1.8). When these are satisfied, there will be corresponding simplifications of the general system of equations (1.1).

2. We first consider the purely elastic case, when $\sigma_0 < \sigma_*$ and $u_0 \sim V_0 t_0$ is very small. For this case $\lambda \equiv 0$ in equation (1.1) and the relation between the stress deviator tensor and the velocity of deformation in system (1.1) may be written in the form

$$G\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right) = \frac{DS_{ij}}{Dt} \tag{2.1}$$

We are able to produce estimates of the different terms in this relation by assuming that

$$v_i = V_0 V_i\left(\frac{x_k}{l}, \frac{t}{t_0}\right), \qquad S_{ij} = \sigma_0 \Sigma_{ij}\left(\frac{x_k}{l}, \frac{t}{t_0}\right) \tag{2.2}$$

where the functions V_i, Σ_{ij} and their derivatives with respect to the dimensionless arguments have order of magnitude unity. We obtain from (2.1)

$$G_0 \frac{V_0}{l} \sim \sigma_0 \left(\frac{1}{t_0} + \frac{V_0}{l}\right) \tag{2.3}$$

Since in this $\sigma_0 \sim \sigma_* \ll G_0$, then from (2.3) it follows that

$$\frac{V_0 t_0}{l} \sim \frac{\sigma_0}{G_0} \ll 1 \tag{2.4}$$

This means that the displacement $u_0 \sim V_0 t_0$ and the deformations u_0/l are small and, in the expressions for the total derivatives with respect to time, one can neglect the convective terms, i.e.

$$\frac{D}{Dt} \sim \frac{d}{dt} \equiv \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} \sim \frac{\partial}{\partial t} \tag{2.5}$$

The system (1.1) is then transformed so that it becomes

$$\begin{gathered} \rho_0 \frac{\partial v_i}{\partial t} = \rho_0 F_i^e - \frac{\partial p}{\partial x_i} + \frac{\partial S_{ij}}{\partial x_j}, \qquad \frac{\partial \theta}{\partial t} + \frac{\partial v_i}{\partial x_i} = 0 \\ p = K\theta, \qquad G_0 \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \frac{\partial v_k}{\partial x_k} \delta_{ij}\right) = \frac{\partial S_{ij}}{\partial t} \end{gathered} \tag{2.6}$$

The second and the fourth of these equations can be integrated and, together with the third, one has Hooke's law

$$S_{ij} = G_0 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{2}{3} \theta \delta_{ij}\right), \quad \theta = -\frac{\partial u_k}{\partial x_k}, \quad p = K\theta \qquad \left(u_i = \int_0^t v_i \, dt\right) \tag{2.7}$$

where u_i are the displacements. In this manner for this case, the system (1.1) goes over into the usual equations for the linear theory of elasticity.

For equation (2.6) it is interesting to consider two basic classes of motion - dynamic and quasi-static. In the first case all the terms in the momentum equation are of the same order

$$\rho_0 \frac{V_0}{t_0} \sim \frac{\sigma_0}{l} \tag{2.8}$$

Comparing this with (2.4) we obtain

$$\frac{l}{t_0} \sim \sqrt{\frac{G_0}{\rho_0}} \sim C, \quad \sigma_0 \sim \rho_0 V_0 C$$

and since $\sigma_0 \sim K\theta \sim G_0\theta$, it follows that $V_0 \sim \theta C$. In these estimates C is the characteristic velocity of an extensional (dilatational) elastic wave. All of these are well-known relations for elastic waves. In the case where the motion is quasi-static, one has

$$\rho_0 \frac{V_0}{t_0} \ll \frac{\sigma_0}{l} \quad \text{or} \quad \frac{l}{t_0} \ll \sqrt{\frac{G_0}{\rho_0}} \sim C \tag{2.9}$$

This condition defines a scale of time t_0 of the boundary functions (1.3) for which elastic waves may be neglected. If one introduces a "wave" time scale $t_w \sim l/C$, then condition (2.9) can be written in the form

$$t_0 \gg t_w \tag{2.10}$$

3. Let us next consider elastic-plastic motion but with small deformations. Here, as in the previous case, $\sigma_0 \sim \sigma_*$, $u_0 \sim V_0 t_0 \ll l$ but $\lambda \neq 0$. It is clear that the estimate of (2.5) still holds. Evaluating approximately the terms in the elasto-plastic flow law

$$G\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right) = \frac{DS_{ij}}{Dt} + \lambda S_{ij}$$

yields

$$G_0 \frac{V_0}{l} \sim \frac{\sigma_0}{t_0} + \left(\frac{G_0 V_0}{\sigma_0 l} + \frac{1}{t_0}\right)\sigma_0 \tag{3.1}$$

On the right-hand side of this relation all of the terms must be of the same order (elastic and plastic components of deformation are of equal order in the case under consideration). This is possible if

$$\frac{G_0 V_0 t_0}{\sigma_0 l} \sim 1 \tag{3.2}$$

Satisfying condition (3.2) reduces immediately to fulfilling condition (3.1) and, further, this condition is reduced to (2.4). Dynamic conditions resulting from these estimates for σ_0 and V_0, and also quasi-static conditions hold just as in the preceding case. System (1.1) goes over into the following:

$$\begin{gathered}
\rho_0 \frac{\partial v_i}{\partial t} = \rho_0 F_i^e - \frac{\partial p}{\partial x_i} + \frac{\partial S_{ij}}{\partial x_j}, \quad p = f(\theta, \theta_*) \\
\frac{\partial \theta}{\partial t} + \frac{\partial v_i}{\partial x_i} = 0, \quad \frac{\partial \theta_*}{\partial t} = \frac{\partial \theta}{\partial t} e\,(\theta - \theta_*)\, e\left(\frac{\partial \theta}{\partial t}\right) \\
G_0\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right) = \frac{\partial S_{ij}}{\partial t} + \lambda S_{ij} \\
\lambda = \frac{2G_0 W - F'(p)\,\partial p/\partial t}{2F(p)} e\,[I_2 - F(p)]\, e\left[2G_0 W - F'(p)\frac{\partial p}{\partial t}\right]
\end{gathered} \tag{3.3}$$

4. We now consider motions with large elastic-plastic deformations, but for moderate stresses, i.e. cases in which $V_0 t_0 \sim l$ and $\sigma_* \ll \sigma_0 \ll G_0$. This is a realistic and practically interesting case because, for example, if $\sigma_* \sim 0.5$ kg/cm^2 and $G_0 \sim 10^3$ kg/cm^2, then we are discussing stresses of the order of $\sigma_0 \sim 10$ kg/cm^2 - 100 kg/cm^2. In this case

$$\frac{G_0 V_0 t_0}{\sigma_0 l} \gg 1 \tag{4.1}$$

and from relation (3.1) (the flow relation) it follows that the elastic components in the flow relation may be neglected. The equation of continuity is transformed into a relation useful for estimating

$$\frac{\theta}{t_0} + \frac{V_0}{l} \sim 0$$

It is easy to show that for $p \sim \sigma_0 \sim 10$ kg/cm^2 - 100 kg/cm^2, $\theta \sim 10^{-2}$. From this and from the previous relation it follows that terms containing derivatives of the density may be neglected. Finally an estimate of the terms in the momentum equation for the dynamic problem yields

$$\rho_0 \left(\frac{V_0}{t_0} + \frac{V_0^2}{l} \right) \sim \frac{\sigma_0}{l} \quad \text{or} \quad \sigma_0 \sim \rho_0 V_0^2 \tag{4.2}$$

Consequently these equations remain invariant (as to their form) with only the substitution of ρ_0 for ρ, of course.

In this way, for this case, from system (1.1) we obtain

$$\begin{gathered}\rho_0 \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = \rho_0 F_i^e - \frac{\partial p}{\partial x_i} + \frac{\partial S_{ij}}{\partial x_j}, \qquad \frac{\partial v_i}{\partial x_i} = 0 \\ \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} = \frac{W}{F(p)} e\,[I_2 - F(p)]\, e(W)\, S_{ij}\end{gathered} \tag{4.3}$$

We observe after this, that with the pressure distribution obtained from (4.3), the problem of finding the density distribution may be solved separately with the help of the relation $p = f(\theta, \theta_*)$ and the functions $\partial p/\partial t + v_i\, \partial p/\partial x_i$.

In the case under consideration, the dynamic condition of motion yields

$$\frac{l}{t_0} \sim \sqrt{\frac{\sigma_0}{\rho_0}} \sim \sqrt{\frac{\sigma_0}{G_0}}\, C \sim \sqrt{\theta}\, C \quad \text{or} \quad t_0 \sim \frac{1}{\sqrt{\theta}}\, t_w \tag{4.4}$$

It is now necessary to note that everything mentioned up to this point has referred only to the region of motion, distant from a wave front. In the region adjoining such a front, the situation will be as in the previous case since from the compatibility condition at the wave front the following estimates on the motion near the front can be made

$$t_0 \sim t_w, \quad V_0 \sim \theta C, \quad \sigma_0 \sim \rho_0 \theta C^2$$

Thus in the region near the front the motion will be described by the system (3.3). It is necessary, however, to bear in mind that in problems for which, in the near-front region, the gradients are large and the wave propagates over distances significantly greater than the wave length (short waves) it is necessary to take into account the convection terms in the derivative with respect to time. In that case the fundamental equations can be significantly simplified by using the work considered in [5] and carrying out a relatively simple analysis which will not be taken up here.

Thus, in the case under consideration, two types of dynamic motion of the medium are possible: in the ante-frontal zone, where the motion has quite strongly a wave character, and far from the front where the wave effects can be neglected and the motion can be described as an incompressible fluid flow. One must observe that the same kind of situations occur in describing capillary motion of fluids (water, for example) which also possess very small compressibility. Thus, for an underwater explosion, the motion is to be understood as quite rapidly dividing into two separate parts with essentially different character in each - pre-frontal motion and motion near the gas bubble; in the first, compressibility is important and the motion propagates in a wave-like manner; in the second, the motion is very well described within the framework of ideal, incompressible fluid theory. Obviously, the reason for this division in the case of a capillary flow and in the case of soils considered here, is the same, namely the small compressibility of the medium.

Although the simplified systems of equations, describing the motion in the pre-frontal region and in the region far from the front, are different, it is possible to form a single unified system which goes over automatically to the correctly corresponding system in one or the other of the two indicated regions. The unified system has the form

$$\begin{gathered}
\rho_0\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = \rho_0 F_i^e - \frac{\partial p}{\partial x_i} + \frac{\partial S_{ij}}{\partial x_j}, \qquad p = f(\theta, \theta_*) \\
\frac{\partial \theta}{\partial t} + \frac{\partial v_i}{\partial x_i} = 0, \qquad \frac{\partial \theta_*}{\partial t} = \frac{\partial \theta}{\partial t} e(\theta - \theta_*) e\left(\frac{\partial \theta}{\partial t}\right) \\
G\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right) = \frac{\partial S_{ij}}{\partial t} + \lambda S_{ij} \qquad (4.5) \\
\lambda = \frac{2GW - F'(p)\,\partial p / \partial t}{2F(p)} e\,[I_2 - F(p)]\, e\left[2GW - F'(p)\frac{\partial p}{\partial t}\right]
\end{gathered}$$

This system differs from the system for the pre-frontal region only in the presence of the convective terms in the total derivative of the velocity with respect to time, so that when these terms are small the system does not, in fact, differ from system (3.3). On the other hand, system (4.5) in the region far from the front, differs but little from system (4.3), since in this region the derivatives with respect to time in all of the equations, except in the momentum equation, are negligibly small.

The quasi-static condition, in the case under consideration, will clearly hold when

$$t_0 \gg \frac{t_w}{\sqrt{\theta}} \gg t_w \tag{4.6}$$

i.e. here we have a stronger condition than in the previous case, for which (2.10) was sufficient. We note that with $\sigma_0 \sim 10 - 100$ kg/cm^2 the value of $\theta \sim 10^{-2}$ and for $l \sim 10$ m and $C \sim 10^2$ m/sec we have $t_w/\sqrt{\theta} \sim$ 1 sec, i.e. the quasi-static condition requires that $t_0 \gg 1$ sec. Thus, for any problem in ordinary structural mechanics of foundations this condition is always satisfied. Only for explosions and shocks should one use the description given by the dynamic equations.

In order to satisfy condition (4.6), the basic equations are obtained from (4.3) in the form

$$\begin{gathered} -\frac{\partial p}{\partial x_i} + \frac{\partial S_{ij}}{\partial x_j} + \rho_0 F_i^e = 0, \qquad \frac{\partial v_i}{\partial x_i} = 0 \\ \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} = \frac{W}{F(p)}\, e\, [I_2 - F(p)]\, e(W)\, S_{ij} \end{gathered} \tag{4.7}$$

These are equations for a rigid-plastic quasi-static flow of soil with developing plastic deformations or, equations of a medium in a "limiting equilibrium" state. In the special case of a planar problem one is able to obtain two-dimensional equations for static equilibrium for a rigid or brittle material [7]. Indeed, from the flow law we have $S_{zz} = 0$, so that the condition of plasticity reduces to the form

$$\tfrac{1}{2}\, [(\sigma_{xx} + p)^2 + (\sigma_{yy} + p)^2 + 2\sigma_{xy}^2] = F(p)$$

and further

$$\sigma_{zz} = -p \quad \text{or} \quad p = -\tfrac{1}{2}\,(\sigma_{xx} + \sigma_{yy})$$

This in its turn reduces to a plasticity condition, finally, in the form

$$\tfrac{1}{4}\, [(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2] = F\, [-\tfrac{1}{2}(\sigma_{xx} + \sigma_{yy})]$$

In the case in which $F(p) = (kp + b)^2$ (see, for example, [3,4]) we have

$$\frac{1}{4}[(\sigma_{xx}-\sigma_{yy})^2+4\sigma_{xy}^2]=\frac{k^2}{4}\left[-(\sigma_{xx}+\sigma_{yy})+2\frac{b}{k}\right]^2$$

In [7], the condition of limiting equilibrium has the form

$$\frac{1}{4}[(\sigma_{xx}-\sigma_{yy})^2+4\sigma_{xy}^2]=\frac{\sin^2\rho}{4}[-(\sigma_{xx}+\sigma_{yy})+2H]^2$$

where ρ is the interior angle of the flow and H is the cohesion .

Thus we see that except for differences in notation these conditions coincide. Supplementing this relation by the equation of equilibrium

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\rho_0F_x^e=0,\qquad \frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\rho_0F_y^e=0$$

we obtain a system of three equations for σ_{xx}, σ_{xy} and σ_{yy} which comprise a unique basis for states of a pulverulent material [7].

It may now be observed that these equations are insufficient for a natural statement and proper solution of problems of statics of a pulverulent material even for the plane case (not even mentioning the fact that the statics of a pulverulent medium generally does not have equations in the spatial case). Secondly, even problems formulated in terms of the stresses are impossible to solve with confidence without constructing the velocity field. An analogous situation existed in the theory of plasticity until a relatively short time ago, when the necessity became obvious of constructing a velocity field and where it was necessary to introduce in the course of the study a specially termed "complete solution" – for describing the solution containing the constructed velocity field. Just as certain solutions of plane problems in the theory of plasticity are unsatisfactory because for these it can be shown that it is impossible to construct a velocity field; one can find such solutions also in statics of pulverulent media.

The derived system (4.7) allows for the study of not only plane problems but all arbitrary spatial problems in the state of limiting equilibrium, and not only problems formulated in terms of the stresses but also problems with any mixed boundary conditions, i.e. any problem with a natural formulation. It must be said, of course, that there exist static problems (and, indeed, dynamic ones in which the main, and perhaps the only, cause for the origination of deformation is the compressibility of the medium (uniaxial compression, for example) and for the solution of these problems it is to be understood that the equations (4.7) are not usable. In these cases it is necessary to begin with the complete equations (1.1) for the quasi-static case.

5. We consider next the case in which the elastic-plastic deformations are large, i.e. $V_0 t_0 \sim l$ and the stresses are large, i.e. $\sigma_0 \sim S_\infty$. If $S_\infty \ll G_0$ then this will be the same as the previous case. If, however, $S_\infty \sim G_0$, then none of the simplifications for the equations (1.1) that have been carried out are possible. This is because in this case θ will not be small: $\theta \sim 1$. The quasi-static condition remains in the form (4.6). For the case of dynamic motion there will not be a subdivision into two types here because with $\theta \sim 1$ motion either near or far from the front must be represented by the complete equations (1.1).

6. Finally, for the case in which one has flow with very large stresses $\sigma_0 \gg S_\infty$ by virtue of which $S_{ij} \sim S_\infty \ll \sigma_0 \sim p$, it is possible to neglect everywhere the tangential stresses, and system (1.1) goes over to the equations for an ideal, compressible fluid with possible irreversible deformations

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = \rho F_i^e - \frac{\partial p}{\partial x_i}, \qquad p = f(\theta, \theta_*)$$
$$\theta = 1 - \frac{\rho_0}{\rho} \qquad \theta_* = 1 - \frac{\rho_0}{\rho_*}, \qquad \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i} = 0 \tag{6.1}$$
$$\frac{\partial \theta_*}{\partial t} + v_i \frac{\partial \theta_*}{\partial x_i} = \left(\frac{\partial \theta}{\partial t} + v_i \frac{\partial \theta}{\partial x_i}\right) e(\theta - \theta_*) e\left(\frac{\partial \theta}{\partial t} + v_i \frac{\partial \theta}{\partial x_j}\right)$$

A consideration of quasi-static motion for this case is not interesting since here it is trivial.

This exhausts the study of all of the essentially different types of soil motion.

It is clear that an analogous analysis can be carried out for any other solid material (for plastic metals, for example) and the results will be completely similar to those obtained here.

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